with

$$C^* \triangleq -I + 2(E + \epsilon_4 R) \tag{16}$$

The inverse of C^* is obtained via Eq. (8), where

$$C = -I + 2(E' + \epsilon_4 \epsilon') \tag{17}$$

Since the transformation in Eq. (8) is orthogonal,

$$C^{-1} = C^T = -I + 2(E' + \epsilon_4 \epsilon'^T) \tag{18}$$

Again, by augmenting ϵ'^T with a fourth row and column, and defining

$$Q \triangleq \begin{bmatrix} \epsilon_4 & -\epsilon_3 & \epsilon_2 & \epsilon_1 \\ \epsilon_3 & \epsilon_4 & -\epsilon_1 & \epsilon_2 \\ -\epsilon_2 & \epsilon_1 & \epsilon_4 & \epsilon_3 \\ -\epsilon_1 & -\epsilon_2 & -\epsilon_3 & \epsilon_4 \end{bmatrix} \tag{19}$$

yields

$$a^* = -b^* + 2(E + \epsilon_4 Q) b^* = C^{*-1} b^* \tag{20}$$

with

$$C^{*-1} \triangleq -I + 2(E + \epsilon_4 Q) \tag{21}$$

Combining Eqs. (16) and (21) yields the relation

$$C^{*-1} = C^* + 4\epsilon_4 \tilde{\epsilon} \tag{22}$$

where $\tilde{\epsilon}$ is the skew-symmetric matrix defined by

$$\tilde{\epsilon} \triangleq \begin{bmatrix} 0 & -\epsilon_3 & \epsilon_2 & 0 \\ \epsilon_3 & 0 & -\epsilon_1 & 0 \\ -\epsilon_2 & \epsilon_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{23}$$

In summary, this note has developed a kinematic transformation using Euler parameters directly and without the necessity of constructing a direction cosine matrix or using quaternion algebra. The transformation pair given by Eqs. (16) and (21) possesses an interesting symmetry and allows for the kinematic transformation of dynamic quantities expressed as four-vectors without having to change their representation back to the three-vectors form.

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Model Reference Adaptive Control for Large Structural Systems

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Introduction

A MODEL reference adaptive control technique as suggested by Sobel et al.[1] has recently been applied to the control of large structural systems.[2,3] In this technique, a reference model incorporating the desirable properties for the plant is selected and, by using command generator tracker procedure[4] and Liapunov stability theory, a control is designed that makes the error vector between the model and plant outputs approach zero asymptotically. The control law is chosen as a linear combination of output errors, reference model states, and reference model inputs, with the adaptive gains being the sum of the integral and proportional gains. Using this procedure, Bar-Kana and co-workers[2] have shown that when collocated actuators and combined position and rate sensors are used, the output error approaches zero, provided the ratio of the position-to-rate output is limited by a function dependent on the damping ratio and the lowest structural frequency.

In this Note, we improve on this limit and show, by constructing a suitable Liapunov function, that the ratio of position-to-rate output is limited by twice the product of the damping ratio and the lowest structural frequency. As well, we show that other control laws (e.g., the relay type) can be designed.

Dynamics of Structures

By using the finite-element method, the dynamics of a linear structure can be represented by the system of differential equations as

$$M\ddot{q} + Kq = \bar{B}u \tag{1}$$

$$y = \bar{C}(\alpha q + \dot{q}) \tag{2}$$

where M is an $(n \times n)$ mass matrix, q an n-vector of coordinates, K an $(n \times n)$ stiffness matrix, $\bar{B}$ an $(n \times m)$ control influence matrix, u and y m-dimensional input and output vectors, respectively, $\bar{C}$ an $(m \times n)$ measurement distribution matrix, and α the weighting factor of position-to-rate measurement.

Let Φ denote the modal matrix, such that

$$\Phi^T M \Phi = I_n = (n \times n) \text{ unit matrix}$$

$$\Phi^T K \Phi = \text{diag}(\omega_1^2, \ldots, \omega_i^2, \ldots, \omega_n^2) \equiv \Omega^2$$

where T denotes the transpose of a matrix. In terms of the modal coordinates $\eta = \Phi^{-1} q$, Eqs. (1) and (2) can be written as

$$\ddot{\eta} + \Omega^2 \eta = \Phi^T \bar{B} u \tag{3}$$

$$y = \bar{C}\Phi(\alpha\eta + \dot{\eta}) \tag{4}$$

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Defining the state vector $x=(\eta_1,\dot{\eta}_1,\ldots,\eta_n,\dot{\eta}_n)^T$ and introducing the damping ratio ζ_i, Eqs. (3) and (4) can be rewritten as

$$\dot{x}=Ax+Bu, B=(0,b_1^T|\ldots|0,b_n^T)^T \tag{5}$$

$$y=Cx=(\alpha c_1,c_1|\ldots|\alpha c_n,c_n)x \tag{6}$$

where b_i is the ith row of $\Phi^T\bar{B}$, c_i the ith column of $\bar{C}\Phi$, $A=\text{diag}\,(A_1,\ldots,A_n)$, and A_i is given by

$$A_i=\begin{bmatrix} 0 & 1 \\ -\omega_i^2 & -2\zeta_i\omega_i \end{bmatrix}$$

Adaptive Control System Design

The objective of the control system is to find, without the explicit knowledge of the plant parameters, a control u such that the output vector y approaches asymptotically the output of the following model:

$$\dot{x}_m=A_mx_m+B_mu_m \tag{7}$$

$$y_m=C_mx_m \tag{8}$$

where x_m is an N_m-vector, u_m and y_m m-vectors, and A_m, B_m, and C_m matrices of appropriate dimensions. Since the plant is theoretically infinite dimensional while the dimension of the reference model must be small for practical implementation, it is necessary to assume that $2n \gg N_m$.

When perfect tracking occurs, i.e., $y=y_m$ for $t\geq 0$, let x^*, u^*, and y^* respectively denote the corresponding plant state, control, and output trajectories. Then

$$\dot{x}^*=Ax^*+Bu^* \tag{9}$$

$$y^*=Cx^*=C_mx_m=y_m \tag{10}$$

In the command generator tracker theory, it is assumed that

$$x^*=S_{11}x_m+S_{12}u_m \tag{11}$$

$$u^*=S_{21}x_m+S_{22}u_m \tag{12}$$

where S_{11}, S_{12}, S_{21}, and S_{22} are matrices of appropriate dimensions. The existence of these matrices is guaranteed under mild restrictions, provided u_m is a step function.[4]

We define the error vector

$$e=x^*-x \tag{13}$$

When $e=0$, i.e., $x^*=x$, we have $y=Cx=Cx^*=C_mx_m=y_m$. Thus, if we find a controller such that $e\to 0$ as $t\to\infty$, we also obtain the desired result, i.e., $y\to y_m$ as $t\to\infty$.

Differentiating Eq. (13) with respect to time, the equation for the error can be written as

$$\dot{e}=Ae+B(u^*-u) \tag{14}$$

Let the controller be defined as a linear combination of the output error vector, reference model state and input vectors, and the integral of the output error vector,

$$u=K_ee_m+K_xx_m+K_uu_m+k\int_0^t e_m\mathrm{d}t \tag{15}$$

where $e_m=y_m-y$ and $k\geq 0$. The adaptive gains K_e, K_x, and K_u will be taken as the sum of proportional gain, relay gain, and integral gain; therefore, we write

$$K_e=K_{ep}+K_{er}+K_{eI} \tag{16}$$

and similar expressions for the other gains K_x and K_u.

Substituting Eq. (15) into Eq. (14) and using Eq. (12), the error equation (14) can be rewritten as

$$\dot{e}=(A-B\bar{K}_eC)e+B(\bar{K}_e-K_e)e_m$$
$$+B(S_{21}-K_x)x_m+B(S_{22}-K_u)u_m-kB\int_0^t e_m\mathrm{d}t \tag{17}$$

where $\bar{K}_e$ is an $(m\times m)$ unspecified matrix.

To determine the expressions for the adaptive gains, we construct the following Liapunov function:

$$V=e^TPe+\text{tr}\,[S_{eI}^{-1}(\bar{K}_e-K_{eI})T_{eI}^{-1}(\bar{K}_e-K_{eI})^T$$
$$+S_{xI}^{-1}(S_{21}-K_{xI})T_{xI}^{-1}(S_{21}-K_{xI})^T$$
$$+S_{uI}^{-1}(S_{22}-K_{uI})T_{uI}^{-1}(S_{22}-K_{uI})^T]$$
$$+2k\int_0^t e_m^T(t')\left[\int_0^{t'} e_m(\tau)\mathrm{d}\tau\right]\mathrm{d}t' \tag{18}$$

where P, S_{eI}, T_{eI}, etc., are all positive definite matrices of appropriate dimensions. Its total time derivative $\dot{V}$ can be written as

$$\dot{V}=e^T[P(A-B\bar{K}_eC)+(A-B\bar{K}_eC)^TP]e$$
$$-2e^TPB\bigg[(K_{ep}+K_{er})e_m+(K_{xp}+K_{xr})x_m$$
$$+(K_{up}+K_{ur})u_m+k\int_0^t e_m\mathrm{d}t\bigg]+2ke_m^T\int_0^t e_m\mathrm{d}t$$
$$+2e^TPB[(\bar{K}_e-K_{eI})e_m+(S_{21}-K_{xI})x_m$$
$$+(S_{22}-K_{uI})u_m]-2\text{tr}\,[S_{eI}^{-1}(\bar{K}_e-K_{eI})T_{eI}^{-1}\dot{K}_{eI}^T$$
$$+S_{xI}^{-1}(S_{21}-K_{xI})T_{xI}^{-1}\dot{K}_{xI}^T+S_{uI}^{-1}(S_{22}-K_{uI})T^{-1}\dot{K}_{uI}^T] \tag{19}$$

By taking

$$B^TP=C \tag{20}$$

$$\dot{K}_{eI}=S_{eI}e_me_m^TT_{eI},\qquad K_{ep}=S_{ep}e_me_m^TT_{ep}$$
$$K_{er}=k_e(\text{sgn}e_m)e_m^TT_{er}\qquad \dot{K}_{xI}=S_{xI}e_mx_m^TT_{xI},$$
$$K_{xp}=S_{xp}e_mx_m^TT_{xp}\qquad K_{xr}=k_x(\text{sgn}e_m)x_m^TT_{xr}$$
$$\dot{K}_{uI}=S_{uI}e_mu_m^TT_{uI},\qquad K_{up}=S_{up}e_mu_m^TT_{up}$$
$$K_{ur}=k_u(\text{sgn}e_m)u_m^TT_{ur} \tag{21}$$

Equation (19) becomes

$$\dot{V}=-e^TQe-2\|e_m\|_1(k_ee_m^TT_{er}e_m$$
$$+k_xx_m^TT_{xr}x_m+k_uu_m^TT_{ur}u_m)$$
$$-2[e_m^TS_{ep}e_me_m^TT_{ep}e_m+e_m^TS_{xp}e_mx_m^TT_{xp}x_m$$
$$+e_m^TS_{up}e_mu_m^TT_{up}u_m] \tag{22}$$

where S_{ep}, T_{ep}, etc., are positive semidefinite matrices, k_e, k_x, k_u are non-negative constants,

$\text{sgn}e_m=(\text{sgn}e_{m1},\ldots,\text{sgn}e_{mm})^T$, $\|e_m\|_1=\sum_{i=1}^m |e_{mi}|$, and

$$P(A-B\bar{K}_eC)+(A-B\bar{K}_eC)^TP=-Q \tag{23}$$

From the recent results of LaSalle (see Ref. 2), it follows that $e_m\to 0$ as $t\to\infty$, provided one of the following conditions holds:

1) Q is positive definite.

2) Q is positive semidefinite and S_{ep}, T_{ep} are positive definite.

3) Q is positive semidefinite and T_{er} is positive definite and $k_e > 0$.

In order to apply the above procedure to the system defined by Eqs. (5) and (6), we consider the case in which the actuators and sensors are collocated, i.e., $\bar{C} = \bar{B}^T$. Let $P = \mathrm{diag}(P_1, P_2, \ldots, P_n)$, where P_i, $i = 1, 2, \ldots, n$ is given by

$$P_i = \begin{bmatrix} 2\alpha\zeta_i\omega_i + \omega_i^2 & \alpha \\ \alpha & 1 \end{bmatrix} \tag{24}$$

Since $\bar{C} = \bar{B}^T$, it follows that $c_i = b_i^T$ and hence Eq. (20) is satisfied.

We now determine conditions under which P is positive definite and Q as defined by Eq. (23) is at least positive semidefinite. Let $\bar{K}_e$ be any positive semidefinite matrix. Then Q can be written as

$$\begin{aligned} Q &= -(PA + A^T P) + 2C^T \bar{K}_e C \\ &= \mathrm{diag}(Q_1, Q_2, \ldots, Q_n) + 2C^T \bar{K}_e C \end{aligned} \tag{25}$$

where Q_i, $i = 1, 2, \ldots, n$ is given by

$$Q_i = 2\begin{bmatrix} \alpha\omega_i^2 & 0 \\ 0 & 2\zeta_i\omega_i - \alpha \end{bmatrix} \tag{26}$$

Assuming that $\omega_i > 0$, $i = 1, 2, \ldots, n$, we obtain from Eqs. (24) and (26) that P_i is positive definite and Q_i is positive semidefinite if $0 \le \alpha \le 2\zeta_i\omega_i$. Hence, P is positive definite and Q is positive semidefinite provided

$$0 \le \alpha \le \beta^* \tag{27}$$

where $\beta^* = \underset{i}{\mathrm{Min}}(\beta_i)$ and $\beta_i = 2\zeta_i\omega_i$ is the ith modal damping. Obviously, Q becomes positive definite if strict inequalities hold in Eq. (27), i.e.,

$$0 < \alpha < \beta^* \tag{28}$$

It should be noted that the bound β^* on α is independent of n and requires, by definition, only the knowledge of the lowest modal damping.

For $\zeta_i = \zeta$, a constant, Eq. (27) reduces to

$$0 \le \alpha \le 2\zeta\omega^* \tag{29}$$

where $\omega^* = \underset{i}{\mathrm{Min}}(\omega_i)$. From Eq. (29), it is easy to see that the limit on α is less restrictive than that obtained in Ref. 2 where it is shown that stability holds provided $\alpha \le \mathrm{Min}(\zeta\omega^*, \xi^{-1}\omega^*)$.

Summary and Conclusions

Implicit model reference adaptive control technique provides a promising approach for the control of large structures. Applying this technique to the collocated case, it is shown that the output error approaches zero asymptotically, provided the weighting factor of position to rate measurement is greater than or equal to zero but less than or equal to the lowest modal damping. The control law in the form of integral, proportional, and relay adaptations along with the integral of the output error is proposed.

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Analysis of the Geometric Dilution of Precision Using the Eigenvalue Approach

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I. Introduction

GEOMETRIC dilution of precision (GDOP) has been discussed[1-3] as a criterion for selecting satellites in the global positioning system (GPS). Previous investigations related to GDOP analysis, were performed by Fang,[1] and Brogan,[2] who attempted to develop simple GDOP calculation methods. They employed an eigenvalue approach, since GDOP can be expressed by eigenvalues of the estimation error covariance matrix. However, they did not clarify the relationship between observation directions and eigenvalues. The benefit of the eigenvalue approach is that the relationship between GDOP and observation directions can be expressed explicitly. By making use of this benefit, this Note proposes a new geometrical interpretation of GDOP by using an eigenvalue approach. Based on this interpretation, two simple methods of selecting GPS satellites are used, which do not need matrix inversion calculations.

II. Geometrical Interpretation of Geometric Dilution of Precision

Based on previous discussions, the estimation-error covariance matrix P can be given by $P = (H^T V^{-1} H)^{-1}$, where $H = (A_1, A_2, \ldots, A_n)^T$ and V is a diagonal matrix whose elements are observation error variances. GDOP is defined by the square root of the trace of the estimation error covariance matrix

$$(\mathrm{GDOP})^2 = \mathrm{tr}(H^T V^{-1} H)^{-1} \tag{1}$$

In the GPS navigation problem, $A_i = (a_i, b_i, c_i, 1)$, where (a_i, b_i, c_i) is the line-of-sight vector from a user to a satellite. In the conventional GDOP analysis, it is assumed that if $V = 1$, then, $(\mathrm{GDOP})^2 = \mathrm{tr}(H^T H)^{-1}$. In the case of considering V, if H' is defined by

$$H' = [(1/\sigma_1)A_1, \ldots, (1/\sigma_n)A_n]^T \tag{2}$$

then $(\mathrm{GDOP})^2 = \mathrm{tr}(H'^T H')^{-1}$. Therefore, the characteristics of GDOP depend on the matrix H or the matrix H'. In the

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